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The separation of Stokes flows

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A study is made of the extent to which local boundary geometry can influence separation in a two-dimensional or an axisymmetric Stokes flow. It is shown that a Stokes flow can separate from a point on a smooth body at an arbitrary angle, which can be determined only by reference to the global solution for the flow past the body, and the dominant mode in the stream function near a point of separation is $O(r^3)$ in the distance r from the separation point. When the body has a protruding cusped edge it is shown that separation can occur at an arbitrary inclination to the edge which must again be determined from the global solution. In this case the stream function is $O(r^{\frac{5}{2}})$ near the edge. When the flow is locally within a wedge-shaped region of angle β, where $\beta \neq \pi$ or 2π, and $\beta > 146{\cdot}3°$, it is shown that the dominant modes near the vertex of the wedge are non-separating modes. It follows that, in general, a Stokes flow around such a wedge cannot separate from the vertex. This conclusion is illustrated by reference to the global solution for uniform axisymmetric flow past a spherical lens, in which the structure of the flow near the rim is examined in detail. In the case of a body having a sharp edge of small but non-zero angle protruding into the flow, so that β is very close to 2π, it is shown that separation occurs exceedingly near to the edge. This happens, for example, in the flow past a thin concave–convex lens, for which separation occurs near the rim on the concave side. The analysis also suggests that a similar separation occurs very near the rim on the flatter side of a thin asymmetric biconvex lens. However, for the symmetric biconvex lens, and, as a special case, the circular disk, no separation occurs on either side near the rim. For $\beta < 146{\cdot}3°$, streaming flow into the vertex of a wedge does not occur because of the presence of an infinite set of vortices, and the possibility of separation at the vertex in the sense discussed here does not arise.

1. Introduction

The biharmonic equation governing the well-known Stokes flow of viscous fluids at very low Reynolds number is an elliptic partial differential equation which requires suitable boundary data to be provided at all points of a closed domain to define a solution to the equation in the interior. Furthermore the solution at any interior point of the domain depends on the data provided at all points around the boundary. In some cases, however, where the shape of the boundary has special features, one may expect the solution of the equation to be particularly influenced by the local boundary conditions imposed by these special features. An interesting example is one in which the boundary has a corner in the two-dimensional sense, at which it suffers a sudden change of slope. It has been shown by Dean & Montagnon (1949) and Moffatt (1964) that the solutions of the equation take on a particular form near such a corner. For solutions in a plane they represent, locally, the solution for flow in a wedge, and the stream function ψ may be written, in plane polar co-ordinates (r, θ), in the form

$$\psi(r,\theta) = r^{n+1}\{A\cos(n+1)\theta + B\sin(n+1)\theta + C\cos(n-1)\theta + D\sin(n-1)\theta\}, \quad (1)$$

where A, B, C and D are constants which are used to satisfy no-slip conditions at each plane, and n is a parameter other than 0, $+1$ or -1. For these values of n the solution must be written in the slightly different forms

$$\psi(r,\theta) = A\cos 2\theta + B\sin 2\theta + C\theta + D \quad (n = -1),$$
$$\psi(r,\theta) = r\{A\cos\theta + B\sin\theta + C\theta\cos\theta + D\theta\sin\theta\} \quad (n = 0),$$
$$\psi(r,\theta) = r^2\{A\cos 2\theta + B\sin 2\theta + C\theta + D\} \quad (n = +1).$$

In the case of axisymmetric flows with a boundary having a discontinuity in slope in the meridian plane, the differential equation for ψ takes on a different mathematical form, but it is clear that the solution near such a corner, where the vertex forms a circular rim about the axis of symmetry, will become essentially two-dimensional in the neighbourhood of the rim, and the effect of the rim curvature is locally a small perturbation on the plane-wedge solution. The same form of solution arises in several other similar situations. The separation of a viscous fluid at the edge of a solid plane boundary into a stress-free surface may be regarded as locally wedge-like, and solutions of the form (1) were used by Michael (1958) to show that the angle of separation cannot take arbitrary values. Michael (1964) also examined the conditions under which the interface between two viscous fluids may become locally sharp. More recently a number of global solutions of the biharmonic equation have revealed that the phenomenon of separation in Stokes flow past obstacles of various shapes is very widespread. The axisymmetric solution given by Dorrepaal *et al.* (1976) for a closed torus placed in a uniform stream reveals separated-flow patterns with systems of closed vortices near the centre of the torus similar to those examined by Moffatt for the case in which n becomes imaginary. Similar behaviour in the axisymmetric flow past two spheres has recently been shown by Davis *et al.* (1976). These flow patterns show that a separation streamline can arise at a point of a smooth solid boundary, at varying angles of inclination to the boundary. The flow in the neighbourhood of these separation points is governed locally by the wedge-type solution given by (1). In the following section the separation at a smooth boundary is examined, and it is seen from the local solution that a separation streamline can detach at any angle from a point on a smooth boundary.

2. Separation at a smooth boundary

In order to describe the local behaviour in this case, we require a solution for ψ in the region $0 \leqslant \theta \leqslant \pi$ occupied by the fluid, satisfying the no-slip conditions $\psi = \partial\psi/\partial\theta = 0$ at the boundaries $\theta = 0, \pi$, and also yielding a separation streamline $\psi = 0$ at some angle α, where $0 < \alpha < \pi$. Satisfaction of the no-slip conditions requires that, with ψ given by (1),

$$A + C = 0, \quad (n+1)B + (n-1)D = 0,$$
$$A\cos(n+1)\pi + B\sin(n+1)\pi + C\cos(n-1)\pi + D\sin(n-1)\pi = 0,$$
$$-A(n+1)\sin(n+1)\pi + B(n+1)\cos(n+1)\pi - C(n-1)\sin(n-1)\pi + D(n-1)\cos(n-1)\pi = 0.$$

The first two of these equations are satisfied when ψ is of the form

$$\psi_n = r^{n+1}\{A_n[\cos(n+1)\theta - \cos(n-1)\theta] + E_n[(n-1)\sin(n+1)\theta - (n+1)\sin(n-1)\theta]\}, \tag{2}$$

where $B_n = (n-1)E_n$ and $D_n = -(n+1)E_n$. The remaining two equations require that

$$\sin(n+1)\pi = 0. \tag{3}$$

We reject real values of $n \leqslant 0$ as representing infinite point-force singularities at $r = 0$. Otherwise, the acceptable solutions of (3) are $n = 2, 3, 4, \ldots$. In the exceptional case $n = 1$ it is easily seen that the only non-trivial solution has the form $\psi_1 = A_1 r^2 \sin^2\theta$, representing the linear shear. A solution representing a separated flow will satisfy $\psi(\alpha) = 0$ at the separation angle $\theta = \alpha$. Thus we see that the local solution does not, in this case, impose any specific angle of separation on any of the modes $n = 2, 3, 4, \ldots$, since α may take any value in each mode by suitable choice of the ratio of A_n to E_n. The solution for $n = 1$ clearly has no separation. We note here that solutions for different values of n are additive. The scaling of the various components which cannot be determined locally reflects the elliptic nature of the problem, and can be determined only by application of the global boundary conditions. Similar remarks apply in this case to the ratio $A_n{:}E_n$ in each mode. We expect in general that the component solutions for $n = 1, 2, 3, \ldots$ will all be present in any global situation at any point of a smooth boundary. Clearly the smaller the value of n the more dominant is the flow for that component near $r = 0$. Points where the $n = 1$ mode is present evidently represent ordinary points of the boundary around which the flow is locally a linear shear. Points of separation occur where the $n = 1$ mode is absent. The strongest separation mode is therefore $n = 2$, with ψ_2 given by

$$\psi_2 = r^3\{A_2(\cos 3\theta - \cos\theta) + E_2(\sin 3\theta - 3\sin\theta)\}.$$

The angle of separation is determined by the ratio $A_2{:}E_2$ imposed by the global solution when this mode is present. In all the global solutions known to the authors, separation at a smooth boundary is always determined by the strongest mode, in this case $n = 2$. This is illustrated in the solutions given by Dorrepaal *et al.* (1976) and Davis *et al.* (1976), where separation lines arise at smooth boundaries. When A_2 or E_2 is zero, ψ_2 is symmetric or antisymmetric about the mid-line $\theta = \frac{1}{2}\pi$, respectively. Separation does not occur when A_2 is zero.

3. Separation at a sharp edge

Another interesting special case occurs in Stokes flow around the edge of a thin plate, which is represented as flow in a wedge of angle 2π. Applying no-slip conditions at $\theta = 0$ and 2π, we find the solution to be given again by (2) with the equation

$$\sin 2\pi n = 0$$

to determine n. Hence $n = \frac{1}{2}, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, \ldots$. For the case $n = 1$ we again find the linear shear solution $\psi_1 = A_1 r^2 \sin^2\theta$. In the range $0 \leqslant \theta \leqslant 2\pi$, ψ_1 gives separation at $\theta = \pi$. However, the dominant mode is now $n = \frac{1}{2}$, giving rise to the stream function

$$\psi_{\frac{1}{2}} = r^{\frac{3}{2}}\{A_{\frac{1}{2}}(\cos\tfrac{3}{2}\theta - \cos\tfrac{1}{2}\theta) + E_{\frac{1}{2}}(\tfrac{1}{2}\sin\tfrac{3}{2}\theta - \tfrac{3}{2}\sin\tfrac{1}{2}\theta)\}.$$

Again, the local conditions are satisfied irrespective of the ratio $A_{\frac{1}{2}}:E_{\frac{1}{2}}$, so that the edge permits separation at any angle. A specific angle can only be determined by reference to the global boundary conditions. An interesting global solution illustrating the separation at a sharp edge is the solution for a spherical cap symmetrically placed in a uniform stream. This was first given by Payne & Pell (1960) for a hemispherical cap as a special case of their treatment for a lens. The solution for a general spherical cap was given by Collins (1963), whose result for a hemispherical cap differed from that of Payne & Pell because of algebraic errors in the Payne & Pell paper. An alternative form of solution to this problem has recently been given by Dorrepaal, O'Neill & Ranger (1976), who have shown that this flow separates at the rim of the cap, and that the angle of separation varies with the angle of the spherical cap. Again this solution near the rim is represented by the strongest mode, $n = \frac{1}{2}$. In the special case in which the cap becomes a circular disk, however, the solution is one in which $A_{\frac{1}{2}}$ is zero and no separation occurs. Another global solution featuring the $n = \frac{1}{2}$ mode without separation was given by Dean (1944) for a cusp-shaped surface in a plane, protruding into a linear shear flow. Both of these examples suggest that no separation occurs when the body has fore-and-aft symmetry relative to the direction of the mainstream. In these cases $\psi_{\frac{1}{2}}$ is clearly a symmetric function about $\theta = \pi$. On the other hand, near the leading edge of a flat plate in a uniform stream at zero incidence, the dominant mode is given by $n = \frac{1}{2}$ with $E_{\frac{1}{2}} = 0$, as was seen by Carrier & Lin (1948). In this case $\psi_{\frac{1}{2}}$ is antisymmetric about $\theta = \pi$.

4. Flow near the vertex of a wedge of arbitrary angle

We now consider the more general problem of the local behaviour in a wedge of angle β, where $0 < \beta < 2\pi$, excluding the two cases $\beta = \pi$, 2π already discussed. Satisfaction of the no-slip conditions on $\theta = \beta$ now requires that

$$A_n \sin n\beta \sin \beta - E_n\{n \cos n\beta \sin \beta - \sin n\beta \cos \beta\} = 0, \tag{4}$$

$$A_n\{n \cos n\beta \sin \beta + \sin n\beta \cos \beta\} + E_n(n^2 - 1) \sin n\beta \sin \beta = 0. \tag{5}$$

In addition, for separation at an angle $\theta = \alpha$ $(0 < \alpha < \beta)$,

$$A_n \sin n\alpha \sin \alpha - E_n\{n \cos n\alpha \sin \alpha - \sin n\alpha \cos \alpha\} = 0. \tag{6}$$

For a non-trivial solution of (4) and (5) we require $\sin^2 n\beta = n^2 \sin^2 \beta$, or, if $x = n\beta$,

$$\sin \beta/\beta = \pm \sin x/x. \tag{7}$$

This is the equation originally studied by Dean & Montagnon (1949) for solutions within a wedge. We are interested here in examining the solutions of these equations for separation at the vertex of the wedge. We note that in general the ratio of A_n to E_n for a given n or x satisfying (7) is no longer arbitrary. Thus, apart from the special cases $\beta = \pi$, 2π, separation can occur only at specific angles α which satisfy (6). Writing $\sin n\beta = \pm n \sin \beta$ in (4) and (5) we have

$$A_n \sin \beta = \pm (\cos n\beta \mp \cos \beta) E_n.$$

Using this equation in (6) the equation for α becomes

$$(\pm \sin n\alpha \sin \alpha \cos n\beta \mp \sin n\alpha \sin \alpha \cos\beta) - \sin \beta(n \cos n\alpha \sin \alpha - \sin n\alpha \cos \alpha) = 0. \tag{8}$$

With the upper choice of sign the stream function is an odd function of $\theta - \frac{1}{2}\beta$, and the flow evidently separates along the line with $\alpha = \frac{1}{2}\beta$. For any β $(< 2\pi)$ for which (7) with

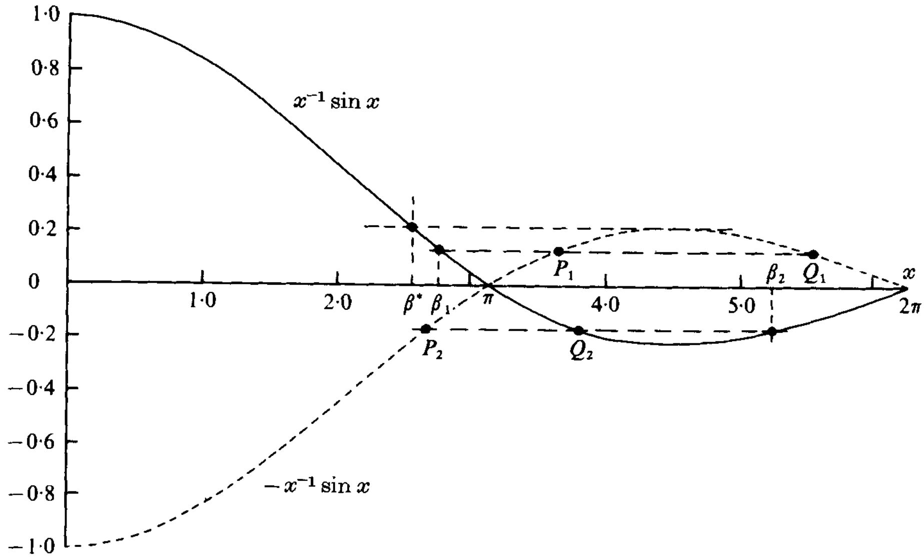

FIGURE 1

the upper sign has a real solution, numerical investigation shows that in the dominant mode there are no other solutions of (8) for which $\alpha < \beta$. With the lower sign the stream function is an even function of $\theta - \frac{1}{2}\beta$, and in the dominant mode in this case there is no solution of (8) for α in which $\alpha < \beta$. Thus separation occurs if the leading mode associated with a solution of (7) with the positive sign is dominant. The pattern of solutions is best seen by reference to the curves of $\pm \sin x/x$ shown in figure 1. Discounting, here, the vortical flow pattern associated with imaginary values of n, we see that solutions with real values of n begin to appear when β reaches $\beta^* = 146{\cdot}3°$ approximately. For $\beta^* < \beta < \pi$, as, for example, at $\beta = \beta_1$ in figure 1, the leading modes are given by the points P_1 and Q_1, both being non-separating modes. The solution represented by P_1 evidently moves into the non-separating linear shear solution as $\beta \to \pi$ with $n = 1$. For $\pi < \beta < 2\pi$, as, for example, at β_2, the dominant mode at P_2 is still a non-separating mode, and the next one, at Q_2, is the first separating mode. Thus we see that for $\beta^* < \beta < 2\pi$ there is no separation at the edge provided that there is a component of the leading mode present in the flow. This has also been shown by Takematsu (1966) in the particular case where $\beta = \frac{3}{2}\pi$. When β reaches 2π, the coalescence of the two solutions at P_2 and Q_2 gives the first opportunity for a separating mode to equal in strength the leading non-separating mode. This coalescence gives rise to the arbitrariness in separation angle which has already been noted when $\beta = 2\pi$ and $n = \frac{1}{2}$. It is worth noting that although $n = 1$ is a solution of (7) with the positive sign for any β, this is a valid solution (other than when $\beta = \pi$ or 2π) only at the stationary points of the curves in figure 1 where $\tan\beta = \beta$.

An illustration of the non-separating flow in a wedge of angle $\beta < 2\pi$ is given by the global solution for axisymmetric flow past lenticular bodies in a uniform stream. We shall show in the next section that for a spherical lens of any non-zero angle at its rim, which excludes the degenerate case of a spherical cap, axisymmetric Stokes flow does not separate at the rim.

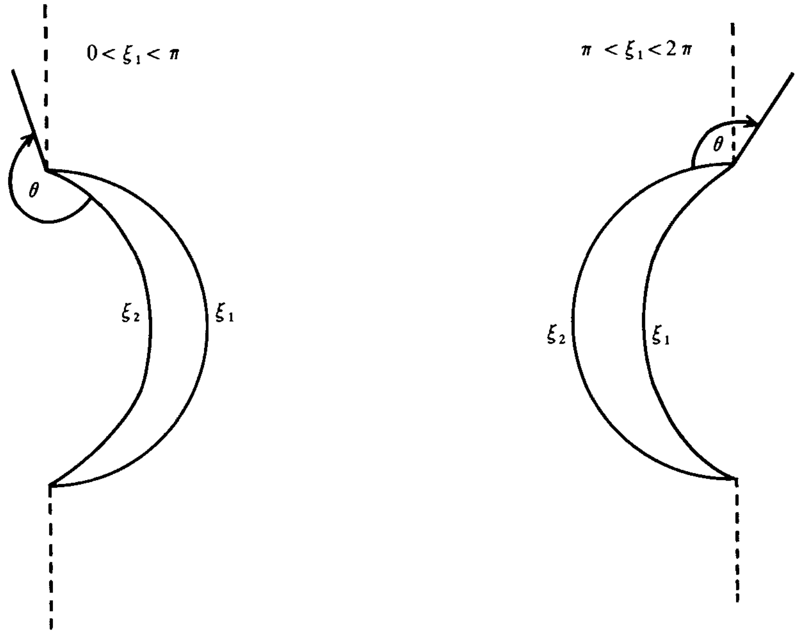

FIGURE 2

5. Flow near the rim of a spherical lens

The Stokes flow solution for a spherical lens placed axisymmetrically in a uniform stream was first given by Payne & Pell (1960), who obtained the stream function in the form

$$\psi = \tfrac{1}{2}U\rho^2\left\{1 - \frac{(s-t)^{\frac{1}{2}}}{[s-\cos(\xi-\xi_0)]^{\frac{1}{2}}} - (s-t)^{\frac{1}{2}}\int_0^\infty \hat{F}(\alpha,\xi)\,K'_\alpha(s)\,d\alpha\right\}, \tag{9}$$

where (ρ, x) are cylindrical polar co-ordinates with $\rho = 0$ along the axis of symmetry of the lens, and U is the mainstream velocity. Peripolar co-ordinates (ξ, η) are defined by

$$x = b\sin\xi/(s-t), \quad \rho = b\sinh\eta/(s-t),$$

where $s = \cosh\eta$, $t = \cos\xi$, and b is a constant. As shown in figure 2, the two surfaces of the lens are defined by $\xi = \xi_1, \xi_2$ and ξ_0 is an arbitrary constant such that

$$0 < \xi_1 < \xi_0 < \xi_2 < 2\pi.$$

The axis of symmetry is given by $\eta = 0$ and the rim of the lens by $\eta = \infty$. The region of flow is $0 < \xi_2 < \xi < 2\pi + \xi_1$. $K_\alpha(s)$ is the conal function $P_{-\frac{1}{2}+i\alpha}(s)$ and $K'_\alpha(s) = dK_\alpha(s)/ds$. The function $\hat{F}(\alpha, \xi)$ is, for general values of ξ_1 and ξ_2, a lengthy expression in α and ξ, which has been found explicitly by Payne & Pell but unfortunately with algebraic errors, and a corrected form for $\hat{F}(\alpha, \xi)$ has up to now not been published. An alternative form for ψ has recently been determined by O'Neill (1976), which avoids the use of the arbitrary constant ξ_0. This solution is given by

$$\psi = 2^{-\frac{1}{2}}U\rho^2(s-t)^{\frac{1}{2}}\int_0^\infty \frac{F(\alpha,\xi)\,K'_\alpha(s)}{(\alpha^2+1)\cosh\alpha\pi}\,d\alpha, \tag{10}$$

where

$$F(\alpha, \xi) = [A(\alpha)\cosh\alpha\xi + B(\alpha)\sinh\alpha\xi - \cosh\alpha(p\pi - \xi)]\cos\xi$$
$$+ [C(\alpha)\cosh\alpha\xi + D(\alpha)\sinh\alpha\xi - \alpha\sinh\alpha(p\pi - \xi)]\sin\xi.$$

In (10), $p = 1$ if $0 \leqslant \xi < 2\pi$ and $p = 3$ if $2\pi \leqslant \xi < 4\pi$. The no-slip conditions are satisfied by choosing $A(\alpha)$, $B(\alpha)$, $C(\alpha)$ and $D(\alpha)$ to make F and $\partial F/\partial\xi$ zero on the boundaries $\xi = \xi_2$ and $2\pi + \xi_1$, so $A(\alpha)$, $B(\alpha)$, $C(\alpha)$ and $D(\alpha)$ depend on ξ_1 and ξ_2. For general values of ξ_1 and ξ_2 they are lengthy expressions and will not be quoted here, since their precise form will not be required in the following analysis. Each of $A(\alpha)$, $B(\alpha)$, $C(\alpha)$ and $D(\alpha)$ is, for all values of ξ_1 and ξ_2, of the form $[\Delta(\alpha)]^{-1}$ times a regular function of α, where

$$\Delta(\alpha) = \alpha^2\sin^2(\xi_2 - \xi_1) - \sinh^2(\xi_2 - \xi_1 - 2\pi)\alpha. \tag{11}$$

The functions $A(\alpha)$ and $C(\alpha)$ are even functions, and $B(\alpha)$ and $D(\alpha)$ are odd functions of α, so that $F(\alpha, \xi)$ is an even function of α. On using the integral representation

$$K_s(\alpha) = \frac{2^{\frac{1}{2}}}{\pi}\coth\alpha\pi\int_\eta^\infty \frac{\sin\alpha u\,du}{(\cosh u - s)^{\frac{1}{2}}}$$

given in Hobson (1931), we may write (10) as

$$2\pi i\psi = U\rho^2(s-t)^{\frac{1}{2}}\frac{d}{ds}\int_\eta^\infty \frac{I(u,\xi)}{(\cosh u - s)^{\frac{1}{2}}}du, \tag{12}$$

where

$$I(u,\xi) = \int_{-\infty}^{\infty} \frac{e^{i\alpha u}F(\alpha,\xi)}{(\alpha^2+1)\sinh\alpha\pi}d\alpha. \tag{13}$$

The integrand of (13) is regular at all points of the α plane except at the zeros of $\Delta(\alpha) = 0$ excluding $\alpha = 0$, and the integral taken around the infinite semicircle in the complex half-plane $\operatorname{Im}\alpha \geqslant 0$ vanishes. The poles in the half-plane $\operatorname{Im}\alpha > 0$ are at $\alpha = i\lambda_m$, $i\bar{\lambda}_m$, $i\mu_m$ and $i\bar{\mu}_m$, where λ_m and μ_m are respectively the roots of the equations

$$\lambda_m\sin(\xi_2 - \xi_1) + \sin(\xi_2 - \xi_1 - 2\pi)\lambda_m = 0 \tag{14}$$

and

$$\mu_m\sin(\xi_2 - \xi_1) - \sin(\xi_2 - \xi_1 - 2\pi)\mu_m = 0, \tag{15}$$

which lie in the first quadrant of the complex α plane, excluding $\alpha = 0$, and with either set of roots arranged in order of increasing real part. At $\alpha = i\lambda_m$ the residue of the integrand of (13) is

$$\frac{e^{-\lambda_m u}F^*(i\lambda_m,\xi)\operatorname{cosec}\lambda_m\pi}{2(\lambda_m^2-1)\lambda_m\sin(\xi_2-\xi_1)[\sin(\xi_2-\xi_1)+(\xi_2-\xi_1-2\pi)\cos(\xi_2-\xi_1-2\pi)\lambda_m]}, \tag{16}$$

where $F^*(\alpha,\xi) = \Delta(\alpha)F(\alpha,\xi)$. The corresponding residue at $\alpha = i\mu_m$ is given by (16) with λ_m replaced by μ_m and the sign of $\xi_2 - \xi_1 - 2\pi$ changed.

In the neighbourhood of the rim, η and s are large and we have, from Hobson,

$$\int_\eta^\infty \frac{e^{-\alpha u}du}{(\cosh u - s)^{\frac{1}{2}}} = 2^{\frac{1}{2}}Q_{\alpha-\frac{1}{2}}(s) \sim c(\alpha)s^{-(\alpha+\frac{1}{2})}$$

for large s, where $c(\alpha) = \pi^{\frac{1}{2}}2^{-\alpha}\Gamma(\alpha+\frac{1}{2})/\Gamma(\alpha+1)$. It follows from (12) that, when $s \gg 1$,

$$\psi \sim -\tfrac{1}{2}Ub^2\operatorname{Re}\left\{\frac{(\lambda_1+2)k_1c(\lambda_1)s^{-(\lambda_1+1)}F^*(i\lambda_1,\xi)\operatorname{cosec}\lambda_1\pi}{(\lambda_1^2-1)\lambda_1\sin(\xi_2-\xi_1)[\sin(\xi_2-\xi_1)+(\xi_2-\xi_1-2\pi)\cos(\xi_2-\xi_1-2\pi)\lambda_1]}\right.$$
$$\left.+\frac{(\mu_1+2)k_2c(\mu_1)s^{-(\mu_1+1)}F^*(i\mu_1,\xi)\operatorname{cosec}\mu_1\pi}{(\mu_1^2-1)\mu_1\sin(\xi_2-\xi_1)[\sin(\xi_2-\xi_1)-(\xi_2-\xi_1-2\pi)\cos(\xi_2-\xi_1-2\pi)\mu_1]}\right\}, \tag{17}$$

where the constants k_1 and k_2 take the values $\frac{1}{2}$ or 1 according as λ_1 or μ_1 are real or complex roots.

It will be noticed that (14) and (15) are equivalent to (7) with n replaced by λ_m or μ_m and $\beta = 2\pi + \xi_1 - \xi_2$, which is the angle at the rim of the flow region bounded by the spherical surfaces of the lens. The studies of Dean & Montagnon and Moffatt therefore show that $\operatorname{Re}\lambda_1 > \operatorname{Re}\mu_1$, except in the degenerate case of a spherical cap, when $\xi_1 = \xi_2$. Thus we may ignore the term in (17) which involves μ_1 with exponentially small error. It is therefore clear that if the angle at the rim is less than $\beta^* \simeq 146{\cdot}3^\circ$, λ_1 is complex and an infinite set of eddies exists in the flow near the rim, but for angles at the rim greater than β^*, λ_1 is real and our earlier discussion of the solutions of (7) shows that the flow is dominated by the non-separating mode with $n = \lambda_1$ and $\lambda_1 > \frac{1}{2}$.

Since b/s measures the distance from a point on the rim when s is large, (17) enables us to see the local behaviour of the stream function near the rim and to corroborate the purely local analysis described earlier. However, to do this, we must specify $F^*(i\lambda_1, \xi)$ and for the purposes of illustration we shall consider the algebraically simpler lens configurations of the symmetrical biconvex lens and the spherical cap.

Symmetrical biconvex lens

In this case the surfaces of the lens are given by $\xi_1 = \xi^*$ and $\xi_2 = 2\pi - \xi^*$, where $0 < \xi^* \leqslant \pi$, and the coefficients $A(\alpha)$, $B(\alpha)$, $C(\alpha)$ and $D(\alpha)$ are found to be

$$\frac{A}{\cosh 2\alpha\pi} = -\frac{B}{\sinh 2\alpha\pi} = \frac{2(\alpha^2\sin^2\xi^* - \sinh^2\alpha\xi^*)\sinh\alpha\pi}{\sinh 2\alpha\xi^* + \alpha\sin 2\xi^*} + \cosh\alpha\pi,$$

$$\frac{C}{\sinh 2\alpha\pi} = -\frac{D}{\cosh 2\alpha\pi} = -\frac{2\alpha(\sin^2\xi^* + \sinh^2\alpha\xi^*)\sinh\alpha\pi}{\sinh 2\alpha\xi^* + \alpha\sin 2\xi^*} + \alpha\cosh\alpha\pi.$$

In terms of local dimensionless polar co-ordinates (r, θ) with origin on the rim and $\theta = 0$ on the surface $\xi = 2\pi - \xi^*$, we have

$$r \sim 1/s, \quad \theta = \xi + \xi^* - 2\pi$$

in the neighbourhood of the rim. Accordingly the local form for ψ is

$$\psi \sim UMb^2r^{\lambda_1+1}\{\bar{A}[\cos(\lambda_1+1)\theta - \cos(\lambda_1-1)\theta] - \bar{B}[(\lambda_1-1)\sin(\lambda_1+1)\theta - (\lambda_1+1)\sin(\lambda_1-1)\theta]\}, \qquad (18)$$

with

$$\bar{A} = (\lambda_1^2 - 1)\sin\xi^*\sin\lambda_1\xi^*, \quad \bar{B} = [\lambda_1\sin^2\xi^* + \sin^2\lambda_1\xi^*]\sin(\lambda_1+1)\xi^*,$$

$$M = \pi^{\frac{1}{2}}\,\Gamma(\lambda_1+\tfrac{3}{2})/2^{\lambda_1}(\lambda_1-1)\,\Gamma(\lambda_1+2)\,(2\xi^*\cos 2\lambda_1\xi^* + \sin 2\xi^*).$$

In the case when $\lambda_1 = 1$, the expression for ψ is interpreted as a limit. It will be noticed that the structure of the solution for ψ near the rim given by (18) is the same as that of (2). By choosing $\xi^* = \pi$, we obtain the solution for axisymmetric streaming flow past a circular disk of radius b. The form of the stream function near the rim is

$$\psi \sim (2^{\frac{1}{2}}U/3\pi)\,b^2r^{\frac{3}{2}}\{3\sin\tfrac{1}{2}\theta - \sin\tfrac{3}{2}\theta\},$$

which represents a non-separating flow for which $\lambda_1 = \frac{1}{2}$. On setting $\xi^* = \frac{1}{2}\pi$, the lens becomes a sphere and the corresponding local form for ψ near the equator is then

$$\psi \sim \tfrac{3}{4}Ur^2\sin^2\theta,$$

which describes a locally non-separating linear shear flow for which $\lambda_1 = 1$.

Spherical cap

In the case of a hemispherical cap, the inner and outer surfaces are $\xi = \frac{1}{2}\pi$ and $\xi = \frac{5}{2}\pi$ and the coefficients appearing in (10) have the simple forms

$$\begin{aligned}
2A\cosh\alpha\pi &= \cosh 2\alpha\pi + (1+2\alpha^2)\cosh\alpha\pi,\\
2B\cosh\alpha\pi &= -\sinh 2\alpha\pi - (1+2\alpha^2)\sinh\alpha\pi,\\
2C\cosh\alpha\pi &= \alpha\sinh 2\alpha\pi - \alpha\sinh\alpha\pi,\\
2D\cosh\alpha\pi &= -\alpha\cosh 2\alpha\pi + \alpha\cosh\alpha\pi.
\end{aligned}$$

Now $\lambda_1 = \frac{1}{2}$ and the form of the solution near the rim is

$$\psi \sim (Ub^2/6\pi)\, r^{\frac{3}{2}}\{3[\cos\tfrac{3}{2}\theta - \cos\tfrac{1}{2}\theta] - [\sin\tfrac{3}{2}\theta - 3\sin\tfrac{1}{2}\theta]\}. \tag{19}$$

This again conforms with the general two-dimensional form of solution given by (2) but is now a separating flow associated with a wedge angle of 2π. Separation occurs from the rim of the cap at an angle of $\theta = 2\tan^{-1}3$, which agrees with the result of Dorrepaal, O'Neill & Ranger (1976) for a hemispherical cap. For a general cap $\xi = \xi_1$, the asymptotic form for ψ corresponding to (19) is

$$\psi \sim (2^{\frac{1}{2}}U/3\pi)\, b^2 \sin^3\tfrac{1}{2}\xi_1\, r^{\frac{3}{2}}\{3[\cos\tfrac{3}{2}\theta - \cos\tfrac{1}{2}\theta]\cot\tfrac{1}{2}\xi_1 - [\sin\tfrac{3}{2}\theta - 3\sin\tfrac{1}{2}\theta]\}, \tag{20}$$

with separation from the rim occurring along the line $\theta = 2\tan^{-1}(3\cot\frac{1}{2}\xi_1)$.

The separation of the flow past a spherical cap at its rim must clearly be regarded as the limiting form of a separation pattern for a thin lens. It may be assumed from these results that a thin lens placed axisymmetrically in a uniform stream has a separated flow pattern on its concave side with a single toroidal vortex adjacent to the axis of symmetry and a point of separation S in the meridian plane on the concave side of the lens. As the lens shrinks to a spherical cap the separation point S moves towards the rim of the lens and in the limit of the spherical cap it reaches the rim. This process is clearly associated with the behaviour of the two modes given by P_2 and Q_2 as $\beta_2 \to 2\pi$ in figure 1. If we write $\beta = 2\pi - 2\epsilon$ and consider ϵ as small, the two values of x in question are $\pi \pm \epsilon$ approximately, for which the values of n are $n = \frac{1}{2} + \epsilon/\pi + O(\epsilon^2)$ and $n = \frac{1}{2} + O(\epsilon^3)$ respectively. Substitution in (4) and (5) shows that in the first of these modes E is $O(\epsilon A)$ and in the second A is $O(\epsilon E)$. The leading terms in the stream function, taking these two modes together, then give

$$\begin{aligned}
\psi = {} & Ar^{\frac{3}{2}+\epsilon/\pi}\{\cos(\tfrac{3}{2}+\epsilon/\pi)\theta - \cos(\tfrac{1}{2}-\epsilon/\pi)\theta + \epsilon[-\tfrac{1}{2}\sin\tfrac{3}{2}\theta + \tfrac{3}{2}\sin\tfrac{1}{2}\theta]\}\\
& + Er^{\frac{3}{2}+O(\epsilon^3)}\{-\tfrac{1}{2}\sin\tfrac{3}{2}\theta + \tfrac{3}{2}\sin\tfrac{1}{2}\theta - \tfrac{3}{4}\epsilon[\cos\tfrac{3}{2}\theta - \cos\tfrac{1}{2}\theta]\},
\end{aligned} \tag{21}$$

where terms in ϵ^2 have been neglected. If we look for values of r at which separation occurs on the boundary $\theta = 2\pi - 2\epsilon$, that is, where $\partial^2\psi/\partial\theta^2 = 0$ at this value of θ, we find the condition

$$r \approx e^{-\frac{7}{3}}(-3\epsilon E/4A)^{\pi/\epsilon - 2}. \tag{22}$$

As $\epsilon \to 0$ the amplitudes A and E of these two modes become of the same order, and the limiting ratio is seen, from (20), to be $A/E = \frac{3}{2}\cot\frac{1}{2}\xi_1$. Thus separation occurs on this boundary where $r \simeq e^{-\frac{7}{3}}(-\frac{1}{2}\epsilon\tan\frac{1}{2}\xi_1)^{\pi/\epsilon - 2}$. This gives a real separation point near the rim provided that $\tan\frac{1}{2}\xi_1 < 0$, i.e. when $\xi_1 > \pi$. This is the condition for the lens surface given by $\xi = \xi_1$ or $\theta = 2\pi - 2\epsilon$ to represent the concave side. When $\xi_1 < \pi$, the concave side is given by $\xi = \xi_2$, which corresponds to $\theta = 0$. It is easily seen from (21) that the point of separation at $\theta = 0$ is then given by

$$r \approx e^{-\frac{7}{3}}\left(\tfrac{1}{2}\epsilon \tan \tfrac{1}{2}\xi_1\right)^{\pi/\epsilon - 2}. \tag{23}$$

The two spherical-cap solutions for $\xi_1 = \pi \pm \gamma$ $(0 < \gamma < \pi)$ clearly represent the same solution, one being obtained from the other by reversal of the flow. It can be seen from (22) and (23) that separation occurs very close to the rim of a thin concave–convex lens. If, for example, we take $\epsilon = 0{\cdot}1$ rad, which corresponds to a lens subtending an angle of about $11{\cdot}5°$ at its rim, for a hemispherical lens in which $\xi_1 = \frac{1}{2}\pi$ the separation radius is of order 10^{-39} from the edge. For a thin lens which is close to the shape of the circular disk we can write $\xi_1 = \pi - \delta$ where $\delta\,(>0)$ is small. Equation (23) then gives $r \approx e^{-\frac{7}{3}}(\epsilon/\delta)^{\pi/\epsilon - 2}$. This formula when applied in the limit as $\epsilon \to 0$ with δ fixed verifies the separation at the rim for a nearly plane spherical cap. If we also assume that it can be applied in the limiting sense in which ϵ and δ approach zero in a fixed ratio it leads us to conjecture that for a very slender asymmetric biconvex lens $r \to 0$ when $\epsilon/\delta < 1$, i.e. that there is a separation point near the rim of the lens on the flatter side. For example, when $\delta = 2\epsilon$ it suggests that separation occurs near the rim on the plane side of a slender plano-convex lens. If we follow this case to the limit as $\epsilon \to 0$, we recover the circular-disk solution in which the separation point comes to the rim of the disk, but the separation region becomes of zero thickness in the limit. When ϵ/δ is fixed at a value greater than unity, (23) shows that $r \to \infty$ as $\epsilon \to 0$, which suggests that there is no separation near the rim of an asymmetrical biconvex lens on the side of greater curvature. When $\epsilon/\delta = 1$ we have the symmetrical biconvex lens with no separation point at or near the rim on either side whatever the angle at the rim. In conclusion we note however that this analysis cannot tell us anything about separation points occurring outside the neighbourhood of the rim of the lens. To find them we must study the global solution of the problem.

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